# Degree conditions restricted to induced paths for hamiltonicity of claw-heavy graphs [*]

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## Abstract

Broersma and Veldman proved that every 2-connected claw-free and $P_6$-free graph is hamiltonian. Chen et al. extended this result by proving every 2-connected claw-heavy and $P_6$-free graph is hamiltonian. On the other hand, Li et al. constructed a class of 2-connected graphs which are claw-heavy and $P_6$-$o$-heavy but not hamiltonian. In this paper we further give some Ore-type degree conditions restricting to induced $P_6$s of a 2-connected claw-heavy graph that can guarantee the graph to be hamiltonian. This improves some previous related results.

**Keywords:** claw-heavy graph; degree condition; hamiltonian graph; closure theory

## 1 Introduction

Throughout this paper, the graphs considered are undirected, finite and simple. For terminology and notations not defined here, we refer the reader to Bondy and Murty [3].

Let $G$ be a graph. For a given graph $H$, we say that $G$ is $H$-free if $G$ contains no induced subgraph isomorphic to $H$. In this case, we call $H$ a *forbidden subgraph* of $G$. Note that if $H_1$ is an induced subgraph of $H_2$, then an $H_1$-free graph is also $H_2$-free.
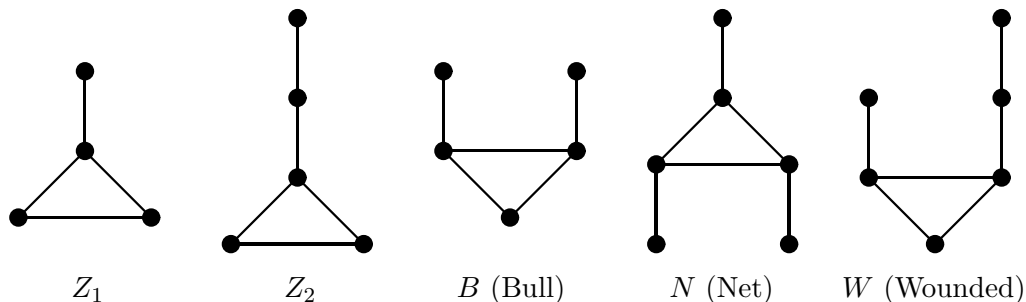
Figure 1: Graphs $Z_1, Z_2, B, N$ and $W$

The bipartite graph $K_{1,3}$ is called the *claw*. Instead of $K_{1,3}$-free, we say that a graph is *claw-free* if it does not contain a copy of $K_{1,3}$ as an induced subgraph. As usual, we use $P_i$ to denote the path of order $i$. Some other special graphs used in this paper are shown in Figure 1.

Forbidden subgraph conditions for hamiltonicity have been studied since the early 1980s, but till 1991, Bedrossian [1] firstly gave a characterization of all pairs of forbidden subgraphs for hamiltonian properties of graphs. First we note that a connected $P_3$-free graph is complete, and clearly is hamiltonian if it has at least three vertices. In fact, it is not difficult to see that $P_3$ is the only connected graph $H$ such that every 2-connected $H$-free graph is hamiltonian. So the following result of Bedrossian deals with pairs of forbidden subgraphs, excluding $P_3$.

**Theorem 1** (Bedrossian [1]). *Let $R, S$ be connected graphs of order at least 3 with $R, S \neq P_3$ and let $G$ be a 2-connected graph. Then $G$ being $R$-free and $S$-free implies $G$ is hamiltonian if and only if (up to symmetry) $R = K_{1,3}$ and $S = P_4, P_5, P_6, C_3, Z_1, Z_2, B, N$ or $W$.*

The above forbidden subgraph conditions for hamiltonicity are sometimes referred to as *structural conditions*. There is another type of conditions with respect to hamiltonian properties of graphs, so-called *numerical conditions*, of which degree conditions may be the most well-known.

Let $G$ be a graph. For a vertex $v \in V(G)$ and a subgraph $H$ of $G$, we use $N_H(v)$ to denote the set, and $d_H(v)$ the number, of neighbors of $v$ in $H$, respectively. We call $d_H(v)$ the *degree* of $v$ in $H$. The *distance* between two vertices $x, y \in V(H)$ in $H$, denoted by $d_H(x, y)$, is the length of a shortest path between $x$ and $y$ in $H$. When no confusion occurs, we will denote $N_G(v)$, $d_G(v)$ and $d_G(x, y)$ by $N(v)$, $d(v)$ and $d(x, y)$, respectively.

The followings are two well-known results concerning the degree conditions for hamiltonicity of graphs.

**Theorem 2** (Dirac [9])**.** *Let $G$ be a graph on $n \geq 3$ vertices. If $d(v) \geq n/2$ for every $v \in V(G)$, then $G$ is hamiltonian.*

**Theorem 3** (Ore [13])**.** *Let $G$ be a graph on $n \geq 3$ vertices. If $d(u) + d(v) \geq n$ for every pair of nonadjacent vertices $u, v \in V(G)$, then $G$ is hamiltonian.*

It is natural to relax the forbidden subgraph conditions to ones in which some of the forbidden subgraphs above are allowed, but some degree conditions are imposed on the subgraphs. Broersma et al. [4] introduced the class of 1-heavy (2-heavy) graphs by restricting Dirac's condition to induced claws of a graph. Instead of Broersma et al.'s restriction, Čada [8] put Ore's condition to induced claws of a graph, and call it an *o*-heavy graph. (In this paper, we will call it a claw-*o*-heavy graph for convenience.) Li et al. [10] extended Čada's concept of claw-*o*-heavy graphs to a more general one.

Let $G$ be a graph on $n$ vertices. Following [10], for a given graph $H$, $G$ is called *H-o-heavy* (the authors used the notation '$H$-heavy' in [10]), if every induced copy of $H$ in $G$ has two nonadjacent vertices with degree sum in $G$ at least $n$. Note that an $H$-free graph is trivially $H$-*o*-heavy, and if $H_1$ is an induced subgraph of $H_2$, then an $H_1$-*o*-heavy graph is also $H_2$-*o*-heavy. Following [11], we say that a graph $G$ is *H-f-heavy* if for every induced copy $G'$ of $H$ in $G$, and every two vertices $u, v \in V(G')$ with $d_{G'}(u,v) = 2$, there holds $\max\{d(u), d(v)\} \geq |V(G)|/2$. Note that every claw-*f*-heavy graph is also claw-*o*-heavy.

Li et al. [10] completely characterized pairs of Ore-type heavy subgraphs for a 2-connected graph to be hamiltonian, which extends Theorem 1. The main result in [10] is given as follows.

**Theorem 4** (Li et al. [10])**.** *Let $R, S$ be connected graphs of order at least 3 with $R, S \neq P_3$ and let $G$ be a 2-connected graph. Then $G$ being $R$-o-heavy and $S$-o-heavy implies $G$ is hamiltonian if and only if (up to symmetry) $R = K_{1,3}$ and $S = P_4, P_5, C_3, Z_1, Z_2, B, N$ or $W$.*

It is easy to see that $P_6$ is the only forbidden subgraph $S$ appearing in Theorem 1 but missing here. Li et al. [10] constructed a class of 2-connected graphs which are claw-*o*-heavy and $P_6$-*o*-heavy but not hamiltonian.

In fact, earlier than Bedrossian [2], Broersma and Veldman [5] proved that every 2-connected claw-free and $P_6$-free graph is hamiltonian. Chen et al. [7] furthermore extended Broersma and Veldman's result as follows.

**Theorem 5** (Chen et al. [7]). *Every 2-connected claw-o-heavy and $P_6$-free graph is hamiltonian.*

So one may ask the question: Which degree conditions can be used to restrict to all induced copies of $P_6$ in a 2-connected claw-$o$-heavy graph to make it hamiltonian?

A related result is as follows.

**Theorem 6** (Ning and Zhang [11]). *Every 2-connected claw-o-heavy and $P_6$-f-heavy graph is hamiltonian.*

One may further ask: Can we still put Ore's condition (or Dirac's condition) to induced $P_6$s of a graph but with some additional restrictions to guarantee that it is hamiltonian?

Our answers are the following two results. Note that the first theorem weakens the condition of Theorem 6.

**Theorem 7.** *Let $G$ be a 2-connected claw-o-heavy graph of order at least $n$. If for every induced copy of $P_6 : v_1v_2\cdots v_6$ in $G$, $d(v_i) + d(v_j) \geq n$ for some $i \in \{1,2,3\}$ and $j \in \{4,5,6\}$, then $G$ is hamiltonian.*

**Theorem 8.** *Let $G$ be a 2-connected claw-o-heavy graph of order at least $n$. If for every induced copy of $P_6 : v_1v_2\cdots v_6$ in $G$, $\max\{d(v_1), d(v_6)\} \geq n/2$, then $G$ is hamiltonian.*

Now we will go further on this direction. Before giving our main results, we will first introduce some necessary terminology and notations.

Let $\gamma$ be a graph (possibly with loops) with vertex set $\mathfrak{I} = \{1, 2, 3, 4, 5, 6\}$ . We say that a graph $G$ is $P_6$-$\gamma$-*heavy* if, for every induced copy of $P_6 : v_1v_2v_3v_4v_5v_6$ in $G$, there exist $i, j \in \mathfrak{I}$ such that $ij \in E(\gamma)$ and $d(v_i) + d(v_j) \geq n$, where $n = |V(G)|$. Note that if $\gamma'$ is a (spanning) subgraph of $\gamma$, then a $P_6$-$\gamma'$-heavy graph is also $P_6$-$\gamma$-heavy.

For two graphs $\gamma$ and $\gamma'$ on $\mathfrak{I}$ such that $ij \in E(\gamma)$ if and only if $(7-i)(7-j) \in E(\gamma')$, we say $\gamma$ is *symmetrical* to $\gamma'$. Note that if $\gamma$ and $\gamma'$ are symmetrical to each other, then a graph $G$ is $P_6$-$\gamma$-heavy if and only if $G$ is $P_6$-$\gamma'$-heavy. If $\gamma$ is symmetrical to itself, then we say $\gamma$ is *symmetrical*.

Let $\varepsilon$ be the empty graph on $\mathfrak{I}$. Then a graph $G$ is $P_6$-free if and only if it is $P_6$-$\varepsilon$-heavy. Let $\sigma$ be the graph on $\mathfrak{I}$ with edge set $E(\sigma) = \{ij : |j - i| \geq 2, i, j \in \mathfrak{I}\}$. Then a graph is $P_6$-$o$-heavy means it is $P_6$-$\sigma$-heavy. Let $\gamma_1$ be the graph on $\mathfrak{I}$ with edge set $\{ij : i = 1, 2, 3 \text{ and } j = 4, 5, 6\}$. Then Theorem 7 states that every 2-connected claw-$o$-heavy and $P_6$-$\gamma_1$-heavy graph is hamiltonian.

The goal of this paper is to find all symmetrical graphs $\gamma$ on $\mathfrak{I}$ such that every 2-connected claw-$o$-heavy and $P_6$-$\gamma$-heavy graph is hamiltonian.
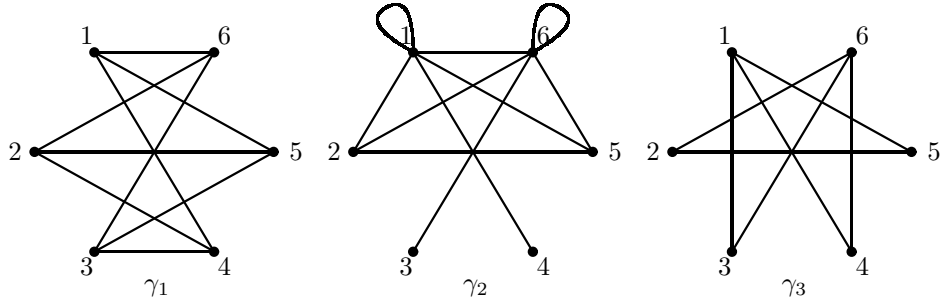
Figure 2: Graphs on $\mathfrak{I}$: $\gamma_1$, $\gamma_2$ and $\gamma_3$

We describe the graphs $\gamma_1, \gamma_2, \gamma_3$ on $\mathfrak{I}$ by giving their edge sets (also see Figure 2.):

$$E(\gamma_1) = \{14, 15, 16, 24, 25, 26, 34, 35, 36\};$$

$$E(\gamma_2) = \{11, 12, 14, 15, 16, 25, 26, 36, 56, 66\};$$

$$E(\gamma_3) = \{13, 14, 15, 25, 26, 36, 46\}.$$

The following is our main result. Note that both Theorems 7 and 8 are its corollaries.

**Theorem 9.** *Let $\gamma$ be a symmetrical graph on $\mathfrak{I}$. Then every 2-connected claw-o-heavy and $P_6$-$\gamma$-heavy graph is hamiltonian if and only if $\gamma$ is a subgraph of $\gamma_1$, $\gamma_2$ or $\gamma_3$.*

## 2   Preliminaries

In this section, we will introduce some preliminaries for the proof of the 'if' part of Theorem 9, which are similar to the ones in [12]. We first introduce Čada's closure theory of claw-$o$-heavy graphs [8], which is an extension of the closure theory of claw-free graphs invented by Ryjáček [14].

Let $G$ be a graph of order $n$. We say that a vertex $x \in V(G)$ is *heavy* in $G$ if $d(x) \geq n/2$; and a pair of vertices $\{x, y\}$ is *heavy* in $G$ if $d(x) + d(y) \geq n$. We say that a vertex (a pair of vertices) is *light* if it is not heavy. Note that if $\{x, y\}$ is a heavy pair, then either $x$ or $y$ is a heavy vertex.

Let $G$ be a graph and $x \in V(G)$. The *local completion(?) of $G$ at $x$*, denoted by $G'_x$, is the graph obtained from $G$ by adding all missing edges in $G[N(x)]$. Define $B_x^o(G) = \{uv : \{u, v\} \subset N(x)$ is a heavy pair of $G\}$. Let $G_x^o$ be the graph with vertex set $V(G_x^o) = V(G)$ and edge set $E(G_x^o) = E(G) \cup B_x^o(G)$. If $G_x^o[N(x)]$ consists of two disjoint cliques $C_1$ and $C_2$, then we call a vertex $z \in V(G) \backslash (\{x\} \cup N(x))$ a *join vertex* of $x$ in $G$ if $\{x, z\}$ is a heavy pair in $G$, and there are two vertices $y_1 \in C_1$ and $y_2 \in C_2$ such that $zy_1, zy_2 \in E(G)$. The

vertex $x$ is an *o-eligible vertex* of $G$, if $N(x)$ is not a clique and, $G_x^o[N(x)]$ is connected or, $G_x^o[N(x)]$ consists of two disjoint cliques and there is some join vertex of $x$.

Let $G$ be a claw-$o$-heavy graph. The *closure* of $G$, denoted by $\mathrm{cl}(G)$, is the graph such that there is a sequence of graphs $G_1, G_2, \ldots, G_t$ and a sequence of vertices $x_1, x_2, \ldots, x_{t-1}$ such that:

(1) $G = G_1$, $G_t = \mathrm{cl}(G)$;

(2) for $i = 1, 2, \ldots, t-1$, $G_{i+1}$ is the local completion(?) of $G_i$ at some $o$-eligible vertex $x_i$ of $G_i$; and

(3) there is no $o$-eligible vertex in $G_t$.

**Theorem 10** (Čada [8]). *Let $G$ be a claw-$o$-heavy graph. Then*

*(1) the closure $\mathrm{cl}(G)$ is uniquely determined;*

*(2) there is a $C_3$-free graph $H$ such that $\mathrm{cl}(G)$ is the line graph of $H$; and*

*(3) $G$ is hamiltonian if and only if $\mathrm{cl}(G)$ is.*

Note that every line graph is claw-free (see [2]). The above theorem implies that $\mathrm{cl}(G)$ is a claw-free graph.

Now we will give some terminology and notations firstly introduced in [12] by the authors. Let $G$ be a claw-$o$-heavy graph and $C$ be a maximal clique of $\mathrm{cl}(G)$. We call $G[C]$ a *region* of $G$. For a vertex $v$ of $G$, we call $v$ an *interior vertex* if it is contained in only one region, and a *frontier vertex* if it is contained in two distinct regions. For two vertices $u, v \in V(G)$, we say $u$ and $v$ are *associated* if $u, v$ are contained in a common region of $G$; otherwise $u$ and $v$ are *dissociated*. We denote by $I_R$ the set of interior vertices of a region $R$, and by $F_R$ the set of frontier vertices of $R$.

From [8], it is not difficult to get the following

**Lemma 1.** *Let $G$ be a claw-$o$-heavy graph. Then*

*(1) every vertex is either an interior vertex of a region or a frontier vertex of two regions;*

*(2) every two regions are either disjoint or have only one common vertex; and*

*(3) every pair of dissociated vertices have degree sum in $\mathrm{cl}(G)$ (and in $G$) less than $|V(G)|$.*

We also need the following tools developed in [12].

**Lemma 2.** *Let $G$ be a claw-$o$-heavy graph and $R$ be a region of $G$. Then*

*(1) $R$ is nonseparable;*

*(2) if $v$ is a frontier vertex of $R$, then $v$ has a neighbor in $I_R$ or $I_R = \emptyset$ and $F_R$ is a clique;*

*(3) for any two vertices $u, v \in V(R)$, there is an induced path of $G$ from $u$ to $v$ such that*

*every internal vertex of the path is in $I_R$; and*

*(4) for two vertices $u, v$ in $R$, if $uv \notin E(G)$ and $\{u, v\}$ is a heavy pair of $G$, then $u, v$ have two common neighbors in $I_R$.*

For two associated vertices $u, v$, by Lemma 2 (3), we use $\Pi[u, v]$ to denote a shortest path such that every internal vertex of it is an interior vertex of the region containing $u, v$. From Lemma 2 (4), we can see that every two vertices of $\Pi[u, v]$ at distance at least 3 in $\Pi[u, v]$ is not a heavy pair in $G$.

Following [6], we define $\mathcal{P}$ to be the class of graphs obtained by taking two vertex-disjoint triangles $a_1 a_2 a_3 a_1$, $b_1 b_2 b_3 b_1$ and by joining every pair of vertices $\{a_i, b_i\}$ by a path $P_{k_i} : a_i c_i^1 c_i^2 \cdots c_i^{k_i-2} b_i$, for $k_i \geq 3$ or by a triangle $a_i b_i c_i$. We denote the graphs in $\mathcal{P}$ by $P_{x_i, x_2, x_3}$, where $x_i = k_i$ if $a_i, b_i$ are joined by a path $P_{k_i}$, and $x_i = T$ if $a_i, b_i$ are joined by a triangle.

The following theorem plays the central role in our proof.

**Theorem 11** (Brousek [6]). *Every non-hamiltonian 2-connected claw-free graph contains an induced subgraph $H \in \mathcal{P}$.*

## 3   Proof of the 'if' part of Theorem 9

Let $G$ be a claw-$o$-heavy non-hamiltonian graph of order $n$. For each $\gamma_k$, $k = 1, 2, 3$, we will show that there exists an induced $P_6 : v_1 v_2 \cdots v_6$ such that for every edge $ij \in E(\gamma_k)$, $d(v_i) + d(v_j) < n$. For convenience, we call such an induced $P_6$ a *bad $P_6$ to $\gamma_k$* in the following.

Let $G' = \mathrm{cl}(G)$. By Theorem 10, $G'$ is claw-free and non-hamiltonian. By Theorem 11, let $H \subseteq G'$ be an induced copy of some graph in $\mathcal{P}$. We denote the vertices of $H$ as in Section 2. If $x_i = k_i$, then let $a_i'$ be the neighbor of $a_i$ on $\Pi[a_i, c_i^1]$, $b_i'$ be the neighbor of $b_i$ on $\Pi[b_i, c_i^{k_i-2}]$, and let $\Pi_i = \Pi[a_i, c_i^1] c_i^1 \Pi[c_i^1, c_i^2] c_i^2 \cdots c_i^{k_i-2} \Pi[c_i^{k_i-2}, b_i]$. If $x_i = T$, then let $a_i'$ be the neighbor of $a_i$ on $\Pi[a_i, c_i]$, $b_i'$ be the neighbor of $b_i$ on $\Pi[b_i, c_i]$, and let $\Pi_i = \Pi[a_i, b_i]$. For $1 \leq i, j \leq 3$, let $\Pi_{ij}^a = \Pi[a_i, a_j]$ and $\Pi_{ij}^b = \Pi[b_i, b_j]$. Let $a_{ij}'$ $(b_{ij}')$ be the neighbor of $a_i$ $(b_i)$ on $\Pi_{ij}^a$ $(\Pi_{ij}^b)$. Set

$$ S = \bigcup_{1 \leq i \leq 3} (\{a_i', b_i'\} \cup \Pi_i) \cup \bigcup_{1 \leq i, j \leq 3} (\Pi_{ij}^a \cup \Pi_{ij}^b). $$

For a path $P$ with the origin $x$, we use $P|_i^x$ (or briefly, $P|_i$) to denote the subpath of $P$ consisting of the first $i$ edges of $P$. If $P = v_1 v_2 \cdots v_p$, then we denote $\overleftarrow{P} = v_p v_{p-1} \ldots v_1$.

**Claim 1.** There is a heavy vertex of $G$ in $S \backslash \{a_i, b_i : 1 \leq i \leq 3\}$, or there are two heavy vertices in $\{a_i, b_i : 1 \leq i \leq 3\}$.

*Proof.* Up to symmetry, suppose that $a_1$ is the vertex with the largest degree among all vertices in $\{a_i, b_i : 1 \leq i \leq 3\}$. If $G$ has no heavy vertex in $S$ or has the only one heavy vertex $a_1$ in $S$, then $P = b_1' b_1 \Pi_{12}^b b_2 \overleftarrow{\Pi_2} a_2 \Pi_{23}^a a_3 a_3'$ is an induced path of order at least 6 and each vertex of $P$ is not heavy in $G$. Thus $P|_5$ is a bad $P_6$ to every $\gamma_k$. $\square$

Note that any two heavy vertices are associated. Up to symmetry, we have the following cases:

**Case 1.** There is a heavy vertex in $\{a_1', b_1'\} \cup (V(\Pi_1) \backslash \{a_1, b_1\})$, or both $a_1$ and $b_1$ are heavy.

Clearly every heavy vertex of $G$ contained in $S$ is in $\{a_1', b_1'\} \cup V(\Pi_1)$. Also clearly either $\bigcup_{1 \leq i,j \leq 3} V(\Pi_{ij}^a)$ or $\bigcup_{1 \leq i,j \leq 3} V(\Pi_{ij}^b)$ contains no heavy pair of $G$. We suppose without loss of generality that $\bigcup_{1 \leq i,j \leq 3} V(\Pi_{ij}^a)$ contains no heavy pair of $G$. Let $Q_1 = a_1' a_1 \Pi_{12}^a a_2 \Pi_2 b_2 \Pi_{23}^b b_3 b_3'$. Then $\overleftarrow{Q_1}|_5$ is a bad $P_6$ to $\gamma_1$.

Suppose now that $\Pi_1 = a_1 x_1 x_2 \cdots x_{p-1} b_1$, where $p$ is the length of $\Pi_1$.

**Case 1.1.** $p = 1$, i.e., $\Pi_1 = a_1 b_1$.

Let $Q_2 = (\Pi_{12}^a a_2 a_2')|_2$ and $Q_2' = (\Pi_{13}^b b_3 b_3')|_2$. Then $\overleftarrow{Q_2} a_1 b_1 Q_2'$ is a bad $P_6$ to $\gamma_2$, $\gamma_3$.

**Case 1.2.** $p = 2$, i.e., $\Pi_1 = a_1 x_1 b_1$.

Let $Q_2 = (\Pi_{12}^b b_2 b_2')|_2$. Then $a_{13}' a_1 x_1 b_1 \overleftarrow{Q_2}$ is a bad $P_6$ to $\gamma_2$, $\gamma_3$.

**Case 1.3.** $p = 3$, i.e., $\Pi_1 = a_1 x_1 x_2 b_1$.

Note that $\{a_1, b_1\}$ is light. Suppose first that $\bigcup_{1 \leq i,j \leq 3} V(\Pi_{ij}^a)$ contains a heavy pair of $G$. Then $b_1$ is heavy, $a_1$ is light, $\{a_1, x_1\}$ is light and $\{a_1, x_2\}$ is light. Let $Q_2 = (b_1 \Pi_{12}^b b_2 b_2')|_2$. Then $a_1 x_1 x_2 b_1 \overleftarrow{Q_2}$ is a bad $P_6$ to $\gamma_5$, $\gamma_2$. Let $Q_3 = x_1 a_1 \Pi_{12}^a a_2 \Pi_2 b_2 \Pi_{23}^b b_3 b_3'$. Then $Q_3|_5$ is a bad $P_6$ to $\gamma_3$.

Now we suppose that $\bigcup_{1 \leq i,j \leq 3} V(\Pi_{ij}^a)$ contains no heavy pairs of $G$. Recall that $\bigcup_{1 \leq i,j \leq 3} V(\Pi_{ij}^b)$ contains no heavy pairs of $G$, either $Q_2 = a_{12}' a_1 x_1 x_2 b_1 b_{13}'$ is a bad $P_6$ to $\gamma_2$, $\gamma_3$.

**Case 1.4.** $p \geq 4$.

If $x_1$ is light, then $Q_2 = (x_1 a_1 \Pi_{12}^a a_2 \Pi_2 b_2 \Pi_{23}^b b_3 b_3')|_5$ contains no heavy vertices of $G$, and hence is bad to $\gamma_2$, $\gamma_3$. So we assume that $x_1$ is heavy, and similarly, $x_{p-1}$ is heavy. This implies that $p = 4$ and $a_1, b_1$ are light.

Note that either $\{a_1, x_1\}$ is light or $\{b_1, x_3\}$ is light. we assume without loss of generality that $\{a_1, x_1\}$ is light. Thus $Q_2 = a_1 x_1 x_2 x_3 b_1 b_{12}'$ is bad to $\gamma_2$, $\gamma_3$.

**Case 2.** There is a heavy vertex in $\bigcup_{1 \leq i,j \leq 3}(V(\Pi_{ij}^a \setminus \{a_i, a_j\})$, or two of $\{a_1, a_2, a_3\}$ are heavy.

Clearly every heavy vertex of $G$ is in $\bigcup_{1 \leq i,j \leq 3} V(\Pi_{ij}^a)$, and at most one of $\{a_i', b_i'\} \cup V(\Pi_i)$ contains heavy pairs of $G$. We assume without loss of generality that both $\{a_1', b_1'\} \cup V(\Pi_1)$ and $\{a_2', b_2'\} \cup V(\Pi_2)$ contain no heavy pairs of $G$.

Let $Q_1 = b_2' b_2 \overleftarrow{\Pi_{12}^b} b_1 \overleftarrow{\Pi_1} a_1 \Pi_{13}^a a_3 a_3'$, then $Q_1|_5$ is a bad $P_6$ to $\gamma_1$.

Suppose now that $\Pi_{12}^a = a_1 x_1 x_2 \cdots x_{p-1} a_2$, where $p$ is the length of $\Pi_{12}^a$.

**Case 2.1.** $p = 1$, i.e., $\Pi_{12}^a = a_1 a_2$.

Let $Q_2 = (a_1' a_1 a_2 \Pi_2 b_2 \Pi_{23}^b b_3 b_3')|_5$. Then $Q_2$ is a bad $P_6$ to $\gamma_2$, $\gamma_3$.

**Case 2.2.** $p = 2$, i.e., $\Pi_{12}^a = a_1 x_1 a_2$.

Let $Q_2 = (a_2' a_2 x_1 a_1 \Pi_1 b_1 b_{13}')|_5$. Then $Q_2$ is a bad $P_6$ to $\gamma_2$, $\gamma_3$.

**Case 2.3.** $p = 3$, i.e., $\Pi_{12}^a = a_1 x_1 x_2 a_2$.

Let $Q_2 = a_1' a_1 x_1 x_2 a_2 a_2'$. Then $Q_2$ is a bad $P_6$ to $\gamma_2$, $\gamma_3$.

**Case 2.4.** $p \geq 4$.

Let $Q_3 = (x_1 a_1 \Pi_1 b_1 \Pi_{12}^b b_2 \overleftarrow{\Pi_2} a_2 x_{p-1})|_5$. Then $Q_3$ is a bad $P_6$ to $\gamma_3$.

If one of $a_1, a_2$ is heavy in $G$, say $a_1$ is heavy, then $x_i$ ($i \geq 3$) and $a_2$ are light. Thus $Q_2 = (a_1' a_1 \Pi_{12}^a)|_5$ is a bad $P_6$ to $\gamma_2$. So we assume that $a_1, a_2$ are light.

Recall that for each two vertices with distance at least 3 in $\Pi_{12}^a$, at least one of them is light. This implies that there exists an integer $i$, $2 \leq i \leq p - 2$, such that every vertex in $V(\Pi_{12}^a) \setminus \{x_{i-1}, x_i, x_{i+1}\}$ is light. Note that either $\{x_{i-2}, x_{i-1}\}$ or $\{x_{i+1}, x_{i+2}\}$ is light (we set $x_0 = a_1$ and $x_p = a_2$). We assume without loss of generality that $\{x_{i-2}, x_{i-1}\}$ is light. Then $Q_2 = (x_{i-2} x_{i-1} \cdots x_{p-1} a_2 a_2')|_5$ is a bad $P_6$ to $\gamma_2$.
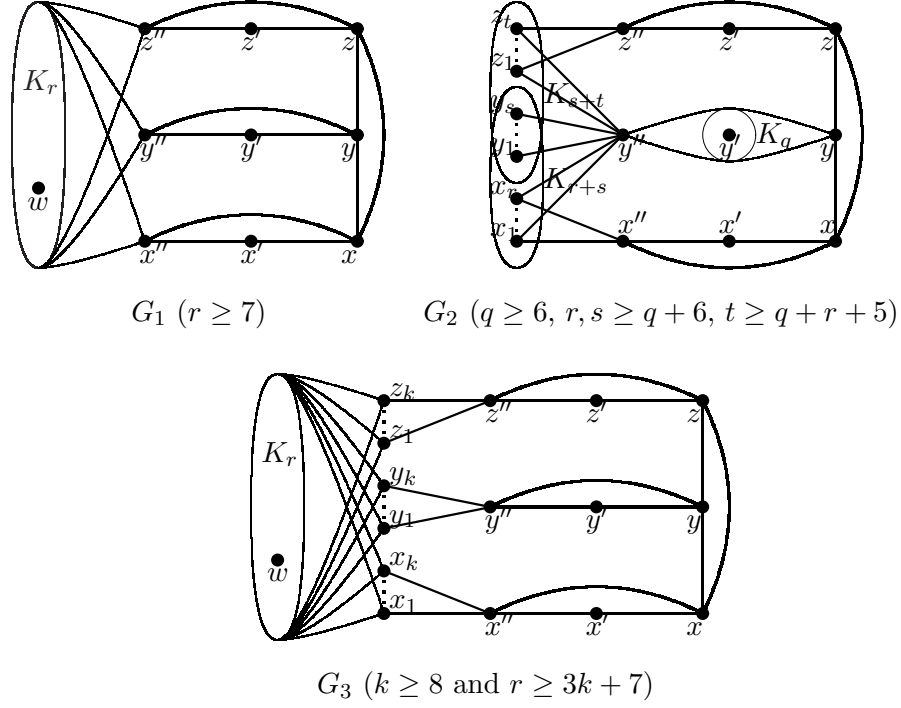
The proof is complete.

Figure 3: Three classes of claw-$o$-heavy non-hamiltonian graphs

## 4 Proof of the 'only if' part of Theorem 9

Let $\gamma$ be a symmetrical graph on $\mathfrak{I}$ such that every 2-connected claw-$o$-heavy and $P_6$-$\gamma$-heavy graph is hamiltonian. We will prove that $\gamma$ is a subgraph of $\gamma_1$, $\gamma_2$ or $\gamma_3$. Assume not. Then for every $k = 1, 2, 3$, $E(\gamma)\backslash E(\gamma_k) \neq \emptyset$. Note that the graphs in Figure 3 are claw-$o$-heavy and non-hamiltonian. Hence they are not $P_6$-$\gamma$-heavy. Let $P = u_1 u_2 \cdots u_6$ and $Q = v_1 v_2 \cdots v_6$ be two induced copies of $P_6$ in a graph $G$ of order $n$. We say $P$ and $Q$ are essentially same if for every $i, j \in [1, 6]$, $d(u_i) + d(u_j) \geq n$ if and only if $d(v_i) + d(v_j) \geq n$.

**Claim 2.** None of $\{22, 23, 24, 33, 34, 35, 44, 45, 55\}$ is in $E(\gamma)$.

*Proof.* Recall that $E(\gamma)\backslash E(\gamma_1) \neq \emptyset$, i.e., one of $\{11, 12, 13, 22, 23, 33, 44, 45, 46, 55, 56, 66\}$ is in $E(\gamma)$. Since $\gamma$ is symmetrical, one of $\{11, 12, 13, 22, 23, 33\}$ is in $E(\gamma)$ and one of $\{44, 45, 46, 55, 56, 66\}$ is in $E(\gamma)$.

Suppose that one of $\{22, 23, 24, 33, 34, 35, 44, 45, 55\}$ is in $E(\gamma)$. Since $\gamma$ is symmetrical, one of $\{22, 23, 24, 33, 34, 44\}$ is in $E(\gamma)$ and one of $\{33, 34, 35, 44, 45, 55\}$ is in $E(\gamma)$. Consider the graph $G_1$. Let $P = v_1 v_2 \cdots v_6$ be an induced path of $G_1$, and let $ij$ be an

edge in $E(\gamma)$ such that

$$ij \in \begin{cases} \{11,12,13,22,23,33\}, & \text{if } P = x'xyy''wz''; \\ \{22,23,24,33,34,44\}, & \text{if } P = xyy''wz''z'; \\ \{33,34,35,44,45,55\}, & \text{if } P = x'x''wy''yz; \\ \{44,45,46,55,56,66\}, & \text{if } P = xx''wy''yzz'. \end{cases}$$

Then $d(v_i) + d(v_j) \geq |V(G_1)|$. Note that $G_1$ has only the four essentially different induced copies of $P_6$. This implies that $G_1$ is $P_6$-$\gamma$-heavy, a contradiction. $\square$

Let $\mathfrak{E}_1 = \{22,23,24,33,34,35,44,45,55\}$. Then for $k = 1,2,3$, $E(\gamma)\backslash(E(\gamma_k)\cup\mathfrak{E}_1) \neq \emptyset$. Note that $E(\overline{\gamma_2})\backslash\mathfrak{E}_1 = \{13,46\}$. Since $\gamma$ is symmetrical, we can see that $13,46 \in E(\gamma)$.

**Claim 3.** None of $\{11,16,66\}$ is in $E(\gamma)$.

*Proof.* Suppose not. Since $\gamma$ is symmetrical, we can see that one of $\{11,16\}$ is in $E(\gamma)$ and one of $\{16,66\}$ is in $E(\gamma)$.

Consider the graph $G_3$. Let $P = v_1v_2\cdots v_6$ be an induced path of $G_3$, and let $ij$ be an edge in $E(\gamma)$ such that

$$ij = \begin{cases} 13, & \text{if } P = wx_1x''xyy'; \\ 11 \text{ or } 16, & \text{if } P = x_1x''xyy''y_1; \\ 46, & \text{if } P = x'xyy''y_1w; \\ 46, & \text{if } P = xyy''y_1wz_1; \\ 46, & \text{if } P = x'x''x_1wy_1y''; \\ 13, & \text{if } P = x''x_1wy_1y''y'; \\ 13, & \text{if } P = x_1wy_1y''yz. \end{cases}$$

Then $d(v_i) + d(v_j) \geq |V(G_3)|$. Note that $G_3$ has only the seven essentially different induced copies of $P_6$. This implies that $G_3$ is $P_6$-$\gamma$-heavy, a contradiction. $\square$

Let $\mathfrak{E}_2 = \mathfrak{E}_1 \cup \{11,16,66\}$. By Claims 2 and 3, $E(\gamma)\backslash(E(\gamma_3)\cup\mathfrak{E}_2) \neq \emptyset$. Note that $E(\overline{\gamma_3})\backslash\mathfrak{E}_2 = \{12,56\}$. Since $\gamma$ is symmetrical, we can see that $12,56 \in E(\gamma)$.

Let $\gamma'$ be a graph on $\mathfrak{I}$ with edge set $E(\gamma') = \{12,13,46,56\}$. Then $\gamma'$ is a subgraph of $\gamma$. Similarly as in Claim 3, one can check that $G_2$ is $P_6$-$\gamma'$-heavy, and then is $P_6$-$\gamma$-heavy, a contradiction. This completes the proof of the 'only if' part of Theorem 9.

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